

IMSC 2058 Solution for Homework 8

Ex 6.1

First, we show that $\|\cdot\|_\infty$ defines a norm on $C(K)$.

By definition, $\|f\|_\infty = \sup_{x \in K} |f(x)| \geq 0$. Moreover, $\|f\|_\infty = 0$ if and only if $\sup_{x \in K} |f(x)| = 0$, which holds if and only if $|f(x)| = 0$ for all $x \in K$, i.e., $f(x) = 0$.

Let c be a scalar. Then

$$\|cf\|_\infty = \sup_{x \in K} |cf(x)| = \sup_{x \in K} |c| \cdot |f(x)| = |c| \cdot \sup_{x \in K} |f(x)| = |c| \cdot \|f\|_\infty.$$

For any $f, g \in C(K)$,

$$\|f + g\|_\infty = \sup_{x \in K} |f(x) + g(x)| \leq \sup_{x \in K} (|f(x)| + |g(x)|) \leq \sup_{x \in K} |f(x)| + \sup_{x \in K} |g(x)| = \|f\|_\infty + \|g\|_\infty.$$

Next, we show the completeness of $C(K)$.

Let $\{f_n\}_{n=1}^\infty \subset C(K)$ be a Cauchy sequence with respect to $\|\cdot\|_\infty$. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_m - f_n\|_\infty < \varepsilon$ whenever $m, n \geq N$.

Fix any $x \in K$. Then $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty < \varepsilon$ for $m, n \geq N$. Thus, $\{f_n(x)\}$ is Cauchy in the complete space \mathbb{R} (or \mathbb{C}), so it converges. Define $f : K \rightarrow \mathbb{R}$ (or \mathbb{C}) by $f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Then $f_n \rightarrow f$ pointwise on K .

Fix $\varepsilon > 0$ and choose N as above. For any fixed $m \geq N$ and all $x \in K$,

$$|f(x) - f_m(x)| = \lim_{n \rightarrow \infty} |f_n(x) - f_m(x)| \leq \sup_{x \in K} |f_n(x) - f_m(x)| = \|f_n - f_m\|_\infty < \varepsilon.$$

Thus f_n converges uniformly to f .

Since each f_n is continuous and $\{f_n\}$ converges uniformly to f on the compact set K , then f is continuous, i.e. $f \in C(K)$. Therefore $C(K)$ is a complete normed space.

Ex 6.3

Let (X, d) be a compact metric space. We will construct a countable dense subset $D \subseteq X$.

Since (X, d) is compact space, then it is totally bounded. That is to say : for every $r > 0$, there are finitely many elements x_1, \dots, x_N in X such that $X = \bigcup_{k=1}^N B(x_k, r)$.

For each positive integer $n \geq 1$, let $r_n = 1/n > 0$. By totally boundedness, there exists a finite subset $F_n \subseteq X$ such that

$$X = \bigcup_{x \in F_n} B\left(x, \frac{1}{n}\right).$$

Note that each F_n is finite and there are countably many n , so we can define a countable set

$$A = \bigcup_{n=1}^{\infty} F_n.$$

To show A is dense in X is to prove that every open set in X contains a point of A .

Fix $y \in X$ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ large enough so that $1/n < \varepsilon$. There exists $x \in F_n \subseteq A$ such that $y \in B(x, 1/n)$, i.e., $d(y, x) < 1/n < \varepsilon$. Thus, we have $x \in B(y, \varepsilon) \cap A$, hence A is dense.